ADITYA ENGINEERING COLLEGE (A)

MULTIPLE INTEGRALS

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Multiple Integrals

1.Double Integral

2.Triple Integral

Multiple integrals is a natural extension of an ordinary definite integral to a function of 2 varibales (double integrals) or 3 variables (Triple integrals)

DOUBLE INTEGRATION

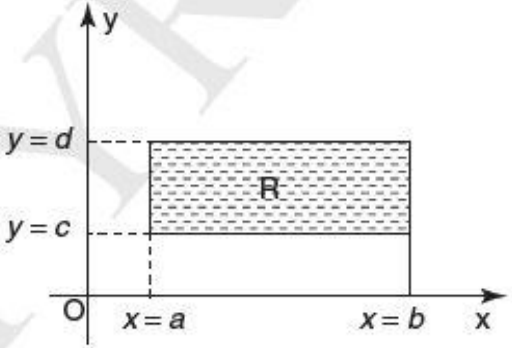
Double integrals occur in many practical problems in science and engineering. It is used in problems involving area, volume, mass, centre of mass.

Evaluation of Double Integrals

In practice, a double integral is computed by repeated single variable integration, integrate with respect to one variable treating the other variable as constant.

Case 1: If the region R is a rectangle given by $R = \{(x, y) / a \le x \le b,\ c \le y \le d\}$ where a, b, c, d are constants, then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\ dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\ dx\right] dy$$



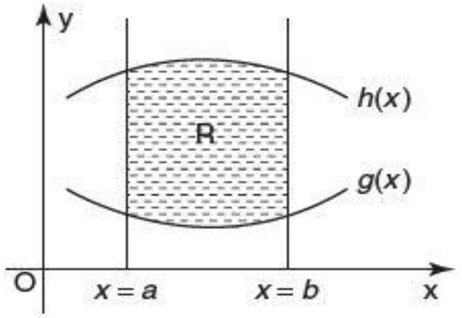
If the limits are constants the order of integration is immaterial, provided proper limits are taken and $f(x, y)$ is bounded in R

Case 2: If the region R is given by

$$R = \{(x, y) / a \le x \le b,\ g(x) \le y \le h(x)\}$$

where a and b are constants, then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\ dy\right] dx$$



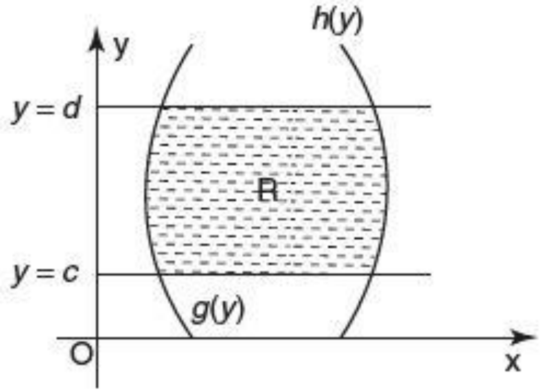
Here the limits of x are constants and the limits of y are functions of x, so we integrate first with respect to y and then with respect to x.

Case 3: If the region R is given by

$$R = \{(x, y) / g(y) \le x \le h(y),\ c \le y \le d\}$$

where c and d are constants then

$$\iint_R f(x, y)\, dx\, dy = \int_c^d \left[\int_{g(y)}^{h(y)} f(x, y)\, dx \right] dy$$



Since the limits of x are functions of y, we integrate first w.r.to x and then w.r.to y.

Note

(1) When variable limits are involved we have to integrate first w.r.to the variable having variable limits and then w.r.to the variable having constant limits.

(2) When all the limits are constants, the order of dx, dy determine the limits of the variable.

Procedure to evaluate double integrals:

Step1: If all the four limits are constants, then follow the given integration order i.e. either "dxdy" or "dydx". Here the inner integration variable takes the limits of inner integral and outer integration varibale takes the limits of outer integral

Step 2: If atleast one of the limits involve variable, then first check whether the integration order is proper or not. Proper order means "checking and following whether "dxdy" or "dydx" order is appropriate according to the limits".

Step 3: Then solve the problem as usual

Step 4: Draw the given region (First use variable limits to draw the region, then use constant limits to complete the region such that their given relationship is preserved.) This sketch helps you to understand the procedure of taking limits

PROBLEMS

1. **Evaluate** $\int_0^1 \int_1^2 x(x+y)\, dy\, dx.$

Solution.

$$\text{Let } I = \int_0^1 \int_1^2 x(x+y)\, dy\, dx = \int_0^1 \left[\int_1^2 x(x+y)\, dy \right] dx$$

$$= \int_0^1 x \left[xy + \frac{y^2}{2} \right]_1^2 dx$$

$$= \int_0^1 x \left\{ \left[x \cdot 2 + \frac{2^2}{2} \right] - \left[x \cdot 1 + \frac{1}{2} \right] \right\} dx$$

$$= \int_0^1 x \left(x + \frac{3}{2} \right) dx = \int_0^1 \left(x^2 + \frac{3x}{2} \right) dx = \left[\frac{x^3}{3} + \frac{3x^2}{2 \cdot 2} \right]_0^1 = \frac{1}{3} + \frac{3}{4} = \frac{13}{12}$$

2. Evaluate $\int_0^1\int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}}$.

Solution.

$$\text{Let } I = \int_0^1\int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \cdot \int_0^1 \frac{dy}{\sqrt{1-y^2}}$$

$$= [\sin^{-1} x]_0^1 [\sin^{-1} y]_0^1 = (\sin^{-1} 1 - \sin^{-1} 0)(\sin^{-1} 1 - \sin^{-1} 0) = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$$

3. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y \, dxdy$.

Solution.

$$\text{Let } I = \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y \, dxdy = \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y \, dydx$$

$$= \int_0^a x^2 \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x^2 (a^2 - x^2) dx$$

$$= \frac{1}{2}\int_0^a (a^2x^2 - x^4) dx = \frac{1}{2}\left[a^2 \frac{x^3}{3} - \frac{x^5}{5}\right]_0^a = \frac{1}{2}\left[a^2 \cdot \frac{a^3}{3} - \frac{a^5}{5}\right] = \frac{1}{2} \cdot \frac{2a^5}{15} = \frac{a^5}{15}$$

Problem

Evaluate $\int_0^2 \int_0^3 xy\,dx\,dy$

Sol:Given,

$$\int_0^2 \int_0^3 xy\,dx\,dy = \int_0^2 [\int_0^3 xy\,dx]dy$$

$$= \int_0^2 [\frac{x^2 y}{2}]_{x=0}^{3} dy = \int_{y=0}^{2} [\frac{9y}{2} - 0]dy$$

$$= \frac{9}{2} \int_{y=0}^{2} y\,dy$$

$$= \frac{9}{2}\left[\frac{y^2}{2}\right]_0^2$$

$$= \frac{9}{4}(4-0) = 9$$

Problem

Evaluate $\int_0^2 \int_0^x y\,dy\,dx$

Sol:Given,

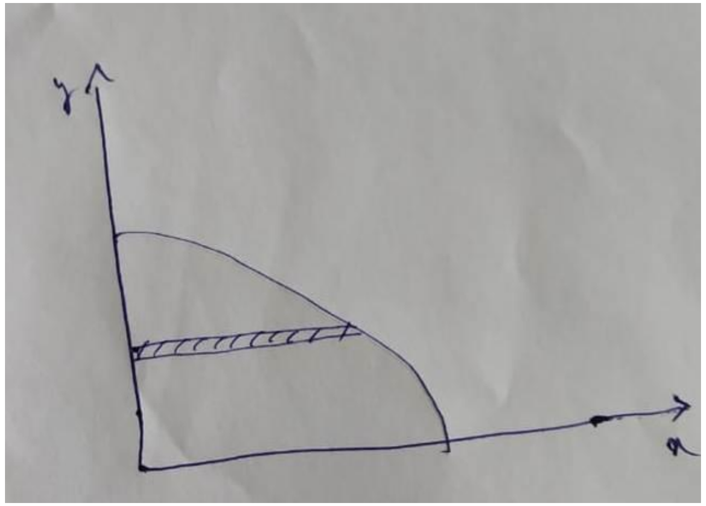
$$\int_{x=0}^{2} \int_{y=0}^{x} y\,dy\,dx = \int_{x=0}^{2} [\int_{y=0}^{x} y\,dy]dx$$

$$= \int_{x=0}^{2} [\frac{y^2}{2}]_{y=0}^{x} dx = \int_{x=0}^{2} [\frac{x^2}{2} - 0]dx$$

$$= \frac{1}{2} \int_{x=0}^{2} x^2 dx$$

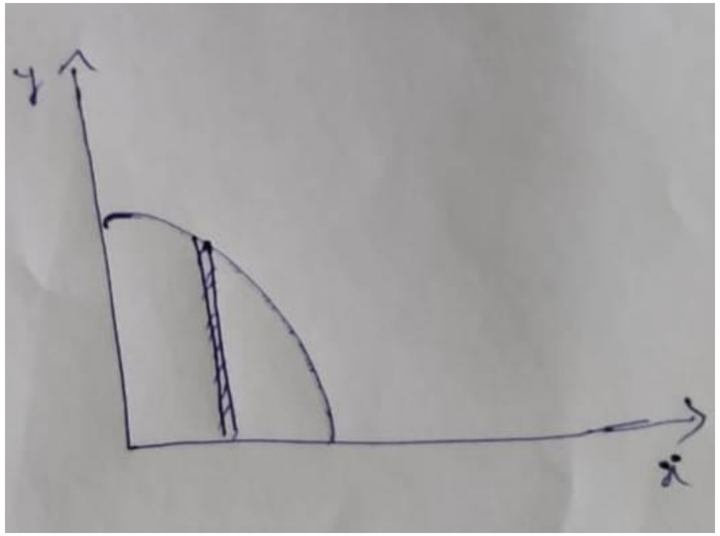
$$= \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2$$

$$= \frac{1}{2} \left(\frac{8}{3} - 0 \right) = \frac{8}{6} = \frac{4}{3}$$



xlimits: Along the **strip** from left to right
Ylimits: Along the **Y-axis** in the rigion of integration

$$\int_{y=a}^{b} \int_{x=f_1(y)}^{f_2(y)} f(x,y)dxdy$$



xlimits: Along the **X-axis** in the region of integration
Ylimits: Along the **strip** from bottom to top

$$\int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} f(x,y)dydx$$

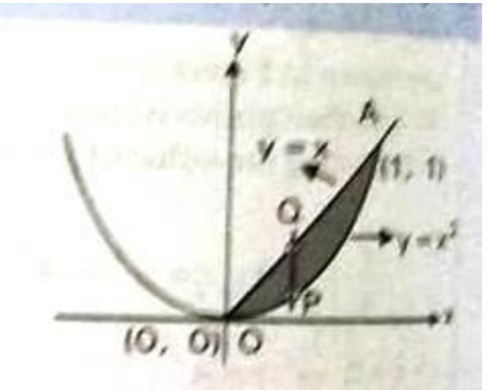
Problem

Evaluate $\iint_R xy(x+y)\,dxdy$ over the region R bounded by $y = x^2$ and $y = x$

Sol: Given R is the region bounded by $y = x^2$ and $y = x$

points of intersection:

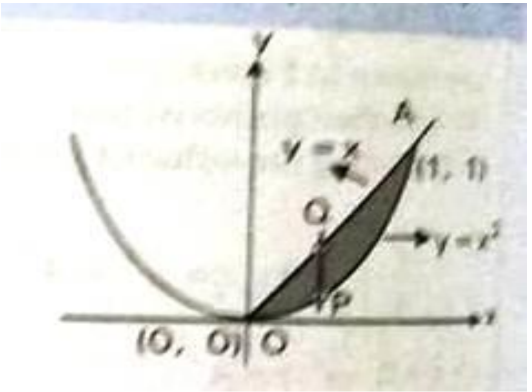
$y = x$ and $y = x^2$



$$x = x^2 \Rightarrow x^2 - x = 0$$
$$\Rightarrow x(x-1) = 0$$
$$\Rightarrow x = 0, x = 1$$
$$x = 0 \Rightarrow y = 0, x = 1 \Rightarrow y = 1$$

The points of intersection are (0,0) &(1,1)

Now consider a vertical strip.



Limits:

y: x^2 to x

x: 0 to 1

$$\iint_R xy(x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_{x=0}^{1}\Big[\int_{y=x^2}^{x}(x^2y+xy^2)\,dy\,dx$$

$$= \int_{x=0}^{1}\Big[\frac{x^2y^2}{2}+\frac{xy^3}{3}\Big]_{y=x^2}^{x}\,dx$$

4. Evaluate $\iint_R xy\ dxdy$ over the positive quadrant of the circle $x^2+y^2=a^2$.

Solution.

Given that the region R is bounded by the coordinate axes $y=0$, $x=0$ and the circle $x^2+y^2=a^2$.

So, the region of integration is the shaded region OAB as in Fig.

To find the limits for x, consider a strip PQ parallel to x-axis, x varies from $x=0$ to $x=\sqrt{a^2-y^2}$.

When we move the strip to cover the region it moves from $y=0$ to $y=a$.

$\therefore$ limits for y are $y=0$ and $y=a$

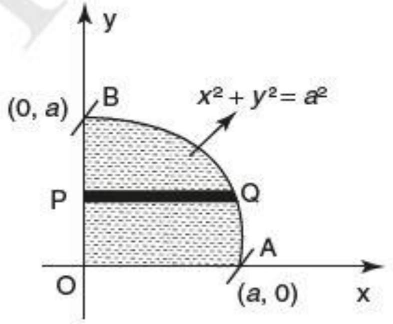
$$\therefore \quad \iint_R xy\ dxdy = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\ dxdy$$

$$= \int_0^a y \cdot \left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a y(a^2-y^2)\,dy$$

$$= \frac{1}{2}\int_0^a (a^2y-y^3)\,dy = \frac{1}{2}\left[a^2\frac{y^2}{2}-\frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[a^2\cdot\frac{a^2}{2}-\frac{a^4}{4}\right] = \frac{1}{2}\cdot\frac{a^4}{4} = \frac{a^4}{8}$$



5. Evaluate $\iint_A xy\ dxdy$, where A is the region bounded by $x = 2a$ and the curve $x^2 = 4ay$.

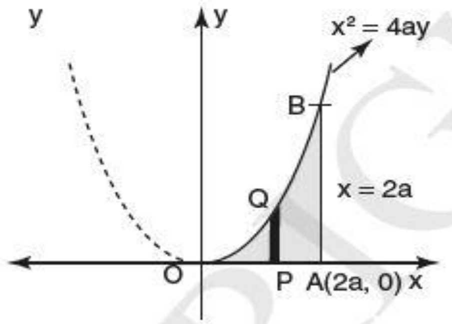
Solution.
Given that the shaded region OAB is the region of integration bounded by $y = 0$, $x = 2a$ and the parabola $x^2 = 4ay$ as in Fig

We first integrate w.r.to y and then w.r.to x.

To find the limits for y, we take a strip PQ parallel to the y-axis, its lower end P lies on $y = 0$ and upper end Q lies on $x^2 = 4ay \Rightarrow y = \frac{x^2}{4a}$

$\therefore$ the limits for y are $y = 0$ and $y = \frac{x^2}{4a}$.

When the strip is moved to cover the area, x varies from $x = 0$ to $x = 2a$.



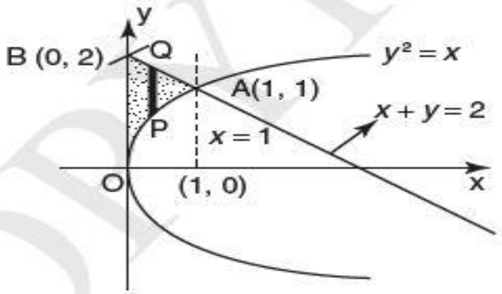
$$\therefore \iint_R xy\ dxdy = \int_0^{2a}\int_0^{\frac{x^2}{4a}} xy\ dydx = \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}} dx$$

$$= \frac{1}{2}\int_0^{2a} x \cdot \frac{x^4}{16a^2}\, dx = \frac{1}{32a^2}\int_0^{2a} x^5 dx = \frac{1}{32a^2}\left(\frac{x^6}{6}\right)_0^{2a} = \frac{1}{32a^2}\left[\frac{2^6 a^6}{6}\right] = \frac{a^4}{3}$$

6. Evaluate $\iint_R x\,dxdy$ over the region R bounded by $y^2 = x$ and the lines $x + y = 2, x = 0, x = 1$.

Solution.

Given that the region of integration is the shaded region OAB as in Fig.



To find A, solve $x + y = 2$ and $y^2 = x$

$\Rightarrow \quad y^2 = 2 - y$

$\Rightarrow \quad y^2 + y - 2 = 0$

$\Rightarrow \quad (y+2)(y-1) = 0 \quad \Rightarrow \quad y = -2, 1$

$\therefore \quad x = 4, 1$

$\therefore$ A is (1, 1) and B is (0, 2) which is the point of intersection of $x = 0$ and $x + y = 2$.

It is convenient to integrate with respect to y first and hence find y limits.

Take a strip PQ parallel to y-axis. P lies on $y^2 = x$ and Q lies on $x + y = 2$.

$\therefore$ the limits for y are $y = \sqrt{x}$ and $y = 2 - x$.

When the strip is moved to cover the region, x varies from 0 to 1.

$$\therefore \iint_R x\,dxdy = \int_0^1 \int_{\sqrt{x}}^{2-x} x\,dydx = \int_0^1 x \cdot [y]_{\sqrt{x}}^{2-x}\,dx$$

$$= \int_0^1 x\left[2 - x - \sqrt{x}\right]dx$$

$$= \int_0^1 (2x - x^2 - x^{3/2})\,dx = \left[2\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^{5/2}}{5/2}\right]_0^1 = 1 - \frac{1}{3} - \frac{2}{5} = \frac{15-5-6}{15} = \frac{4}{15}$$

Change of Order of Integration

The double integral with variable limits for y and constant limits for x is $\int_b^a \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx$. To evaluate this integral, we integrate first w.r.to y and then w.r.to x. This may sometimes be difficult to evaluate. But change in the order of integration will change the limits of y from c to d where c and d are constants and the limits of x from $g_1(y)$ to $h_1(y)$. The double integral becomes $\int_c^d \int_{g_1(y)}^{h_1(y)} f(x, y)\, dx\, dy$ and hence the evaluation may be easy. To evaluate this integral, we integrate first w.r.to x and then w.r.to y.

This process of changing a given double integral into an equal double integral with order of integration changed is called **Change of order of integration**.

For doing this we have to identify the region R of integration from the limits of the given double integral. Sometimes this region R may split into two regions R_1 and R_2 when we change the order of integration and hence the given double integral $\iint_R f(x, y)\, dx\, dy$ will be the sum of two double integrals.

i.e.,

$$\iint_R f(x, y)\, dx\, dy = \iint_{R_1} f(x, y)\, dx\, dy + \iint_{R_2} f(x, y)\, dx\, dy$$

Change of Order of Integration

If the integral is given as $\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx$

Draw the region of integration by drawing the curves $y = f_1(x), y = f_2(x)$ and the lines x=a, x=b .Now consider the horizontal strip so that first we will get the x-limits along the strip in terms of y and the limits for y as constants. Hence the order of integration is changed

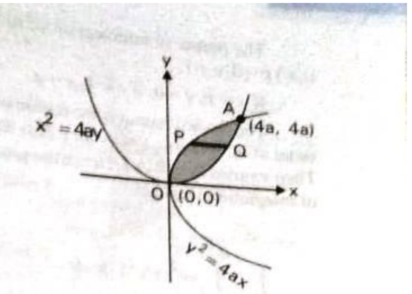
Similarly, the order of integration for $\int_{y=a}^{b}\int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$ is changed

Problem

Change the order of integration and evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$

Sol: Given R is the region bounded by $y = \frac{x^2}{4a}$, $y = 2\sqrt{ax}$

i.e., $x^2 = 4ay,\ y^2 = 4ax$



points of intersection :

$$x^2 = 4a(2\sqrt{ax}) \Rightarrow x^4 = 16a^2 \times 4ax \Rightarrow x(x^3 - 4^3a^3) = 0$$

$$\Rightarrow x = 0, x = 4a$$

$$x = 0 \Rightarrow y = 0, x = 4a \Rightarrow y = 4a$$

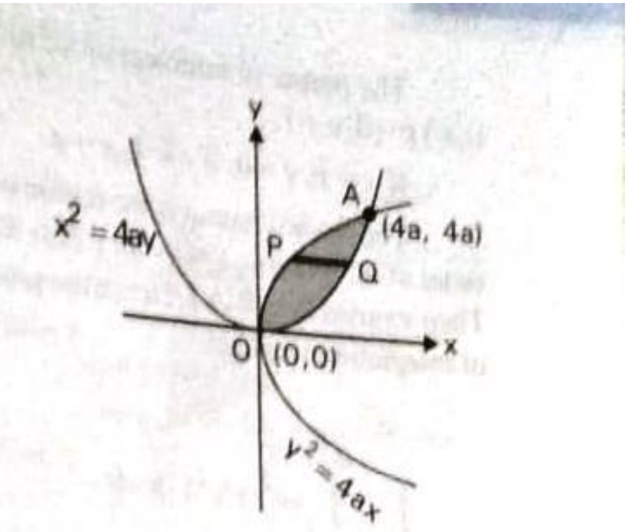
The points of intersection are (0,0) &(4a,4a)

To change the order of integration, consider the horizontal strip.

Limits:

x: $\frac{y^2}{4a}$ to $2\sqrt{ay}$

y: 0 to 4a



$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx = \int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dxdy$$

$$= \int_{y=0}^{4a}\left[\int_{x=y^2/4a}^{2\sqrt{ay}} dx\right]dy$$

$$= \int_{y=0}^{4a}[x]_{x=y^2/4a}^{2\sqrt{ay}}\, dy$$

$$= \int_{y=0}^{4a} [2\sqrt{ay} - y^2/4a]dy$$

$$= \int_{y=0}^{4a} [2\sqrt{a}\,y^{1/2} - \frac{y^2}{4a}]dy$$

$$= [\frac{2\sqrt{a}\,y^{3/2}}{(3/2)} - \frac{y^3}{12a}]_0^{4a}$$

$$= \frac{2a^{1/2}(4a)^{3/2}}{3/2} - \frac{64a^3}{12a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

7. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ by changing the order of integration.

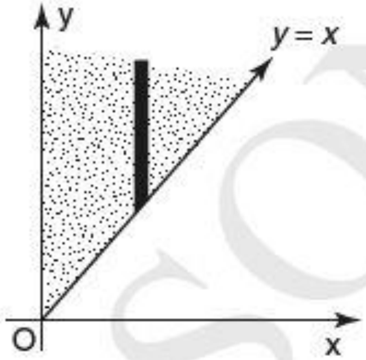
Solution.

Let $$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$$

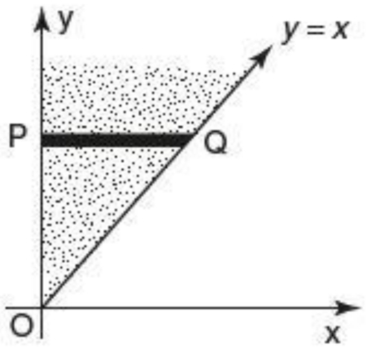
The region of integration is bounded by $y = x$, $y = \infty$, $x = 0$, $x = \infty$.
$\therefore$ the region is unbounded as in Fig.

In the given integral, integration is first with respect to y and then w.r.to x.

After changing the order of integration, first integrate w.r.to x and then w.r.to y. To find the limits of x, take a strip PQ parallel to x-axis (see Fig. 13.10) with P on the line $x = 0$ and Q on the line $x = y$ respectively.



Given order of integration



After the change of order of integration

$\therefore$ the limits of x are $x = 0$ and $x = y$ and the limits of y are $y = 0$ and $y = \infty$

$$\therefore\ I = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty \frac{e^{-y}}{y} \cdot [x]_0^y\, dy = \int_0^\infty \frac{e^{-y}}{y} \cdot y\, dy = \int_0^\infty e^{-y}\, dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty = -(e^{-\infty} - e^0) = -(0-1) = 1$$

8. Evaluate by changing the order of integration $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy dx$.

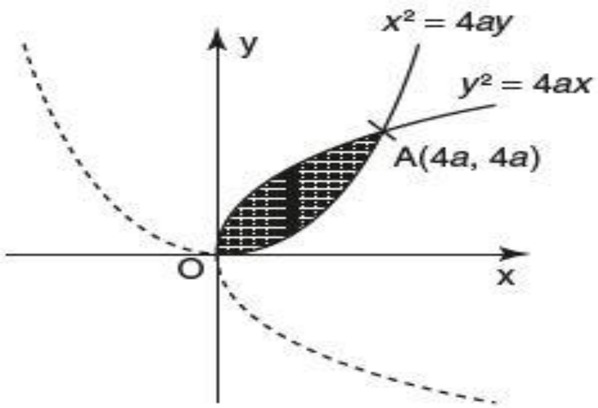
Solution.

Let $I = \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy dx$

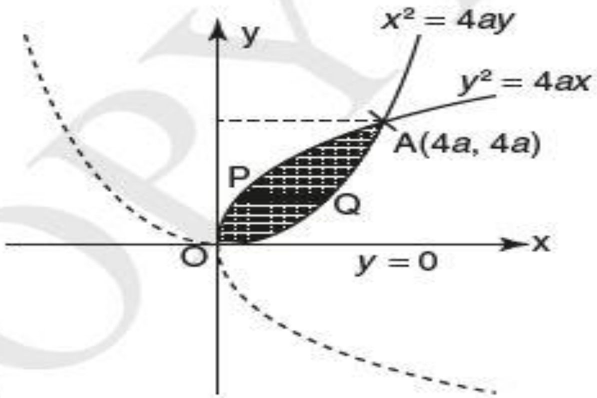
The region of integration is bounded by $y = \frac{x^2}{4a}$, $y = 2\sqrt{ax}$ and $x = 0$, $x = 4a$.

$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay$ is a parabola and $y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$ is a parabola.

In the given integral, integration is first w.r.to y and then w.r.to x. After changing the order of integration, we have to integrate first w.r.to x and then w.r. to y.



Given order of integration



After the change of order of integration

To find the points of intersection of the curves $x^2 = 4ay$ and $y^2 = 4ax$, solve the two equations.

$$x^4 = 16a^2y^2 = 16a^2 \cdot 4ax = 64a^3x$$

$$\Rightarrow \quad x(x^3 - 64a^3) = 0 \quad \Rightarrow \quad x = 0 \quad \text{and} \quad x^3 - 64a^3 = 0$$

Now $\quad x^3 - 64a^3 = 0 \quad \Rightarrow \quad x^3 = 64a^3 = (4a)^3 \quad \Rightarrow \quad x = 4a$

When $x = 0$, $y = 0$ and when $x = 4a$, $y = \dfrac{x^2}{4a} = \dfrac{16a^2}{4a} = 4a$

Points of intersection are O(0, 0) and A is $(4a, 4a)$

Now to find the x limits, take a strip PQ parallel to the x-axis (see Fig. 13.11) where P lies on $y^2 = 4ax$ and Q lies on $x^2 = 4ay$.

$\therefore$ the limits of x are $x = \dfrac{y^2}{4a}$ and $x = 2\sqrt{ay}$

When the strip is moved to cover the region, y varies from 0 to $4a$.

$$\therefore \quad I = \int_a^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{a}\sqrt{y}} dx dy = \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{a}\sqrt{y}} dy$$

$$= \int_0^{4a} \left[2\sqrt{a}\sqrt{y} - \frac{y^2}{4a}\right] dy$$

$$= \int_0^{4a} \left[2a^{1/2}y^{1/2} - \frac{y^2}{4a}\right] dy$$

$$= \left[2a^{1/2}\frac{y^{3/2}}{3/2} - \frac{1}{4a}\frac{y^3}{3}\right]_0^{4a} = \frac{4a^{1/2}}{3}(4a)^{3/2} - \frac{1}{4a}\frac{(4a)^3}{3} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

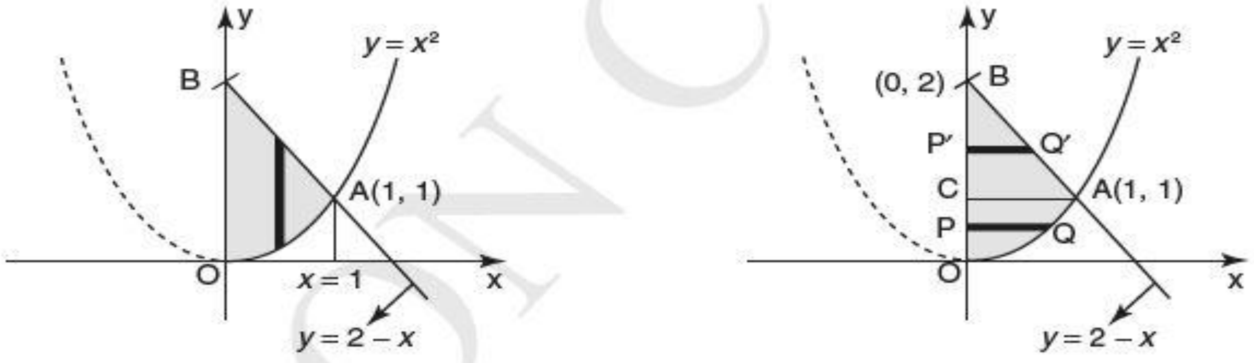
9. Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\, dydx$ and hence evaluate.

Solution.

Let $I = \int_0^1 \int_{x^2}^{2-x} xy\, dydx$

The region of integration is bounded by $x = 0, x = 1, y = x^2, y = 2 - x$.

In the given integral, first integrate with respect to y and then w.r.to x. After changing the order we have to first integrate w.r.to x, then w.r.to y.



Given order of integration

After the change of order of integration

To find A, solve $y = x^2, \quad y = 2 - x$

$\Rightarrow \quad x^2 = 2 - x \quad \Rightarrow \quad x^2 + x - 2 = 0 \quad \Rightarrow \quad (x+2)(x-1) = 0 \quad \Rightarrow \quad x = -2, 1$

Since the region of integration is OAB, $x = 1 \Rightarrow y = 1$

$\therefore$ A is (1, 1) and B is (0, 2), which is the point of intersection of y-axis $x = 0$ and $y = 2 - x$

Now to find the x limits, take a strip parallel to the x-axis. We see there are two types of strips PQ and P′Q′ after the change of order of integration (see Fig. 13.16) with right end points Q and Q′ are respectively on the parabola $y = x^2$ and the line $y = 2 - x$. So, the region OAB splits into two regions OAC and CAB as in Fig.

Hence, the given integral I is written as the sum of two integrals

In the region OAC, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1

In the region CAB, x varies from 0 to $2 - y$ and y varies from 1 to 2

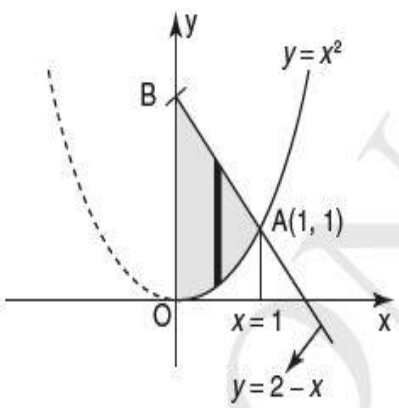
$$\therefore \quad I = \iint_{OAB} xy\,dxdy = \iint_{OAC} xy\,dxdy + \iint_{CAB} xy\,dxdy$$

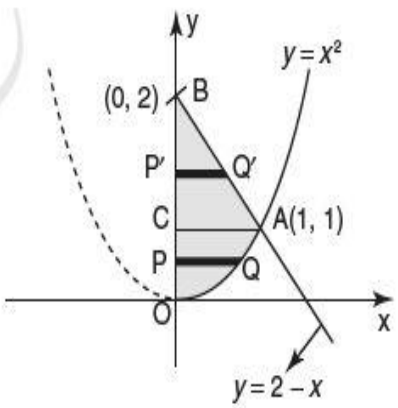
$$= \int_0^1 \int_0^{\sqrt{y}} x\,y\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y \cdot \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y \left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y\,y\,dy + \frac{1}{2}\int_1^2 y \cdot (2-y)^2\,dy$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(4 - 4y + y^2)dy$$





$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(4-4y+y^2)dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y-4y^2+y^3)dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[4\frac{y^2}{2} - 4\frac{y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[2(2^2-1^2) - \frac{4}{3}(2^3-1^3) + \frac{1}{4}(2^4-1^4)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[6 - \frac{4}{3}\times 7 + \frac{1}{4}\times 15\right] = \frac{1}{6} + \frac{1}{2}.\frac{[72-112+45]}{12} = \frac{1}{6} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8}$$

Double Integral in Polar Coordinates

To evaluate the double integral of $f(r, \theta)$ over a region R in polar coordinates, generally we integrate first w.r.to r and then w.r.to θ. So, the double integral is $\int_{\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} f(r,\theta)\, dr d\theta$

However, whenever necessary, the order of integration may be changed with suitable changes in the limits. As in Cartesian, when we integrate w.r.to r, treat θ as constant.

NOTE

1. $\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \dots \frac{2}{3} \cdot 1$ if n is odd and $n \geq 3$

2. $\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \dots \frac{1}{2} \cdot \frac{\pi}{2}$ if n is even

3. $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$ if $n \neq -1$

10. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\cos\theta} r^2 dr d\theta$.

Solution.

$$\text{Let } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\cos\theta} r^2 dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^3\theta d\theta$$

$$= \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta d\theta = \frac{8}{3}\cdot 2\int_{0}^{\frac{\pi}{2}} \cos^3\theta d\theta$$

[$\because \cos^3\theta$ is an even function]

$$= \frac{16}{3}\cdot\frac{2}{3}\cdot 1 = \frac{32}{9}$$

[Using formula]

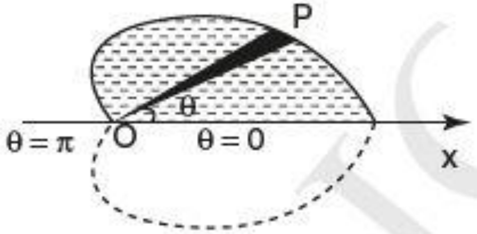
11. Evaluate $\iint_A r\sin\theta \, dr d\theta$ over the area of the cardioid $r = a(1 + \cos\theta)$ above the initial line.

Solution.

Let $I = \iint_A r\sin\theta dr d\theta$

First integrate w.r.to r.

Take a radial strip OP, its ends are on $r = 0$ and $r = a(1 + \cos\theta)$. When it is moved to cover the area, θ varies from 0 to π



$$\therefore \quad I = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r\sin\theta \, dr d\theta = \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} \sin\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2 \sin\theta d\theta$$

$$= -\frac{a^2}{2}\int_0^{\pi}(1+\cos\theta)^2(-\sin\theta)d\theta$$

$$= -\frac{a^2}{2}\left[\frac{(1+\cos\theta)^3}{3}\right]_0^{\pi} \qquad \left[\because \frac{d}{d\theta}(1+\cos\theta) = -\sin\theta\right]$$

$$= -\frac{a^2}{6}\left[(1+\cos\pi)^3 - (1+\cos 0)^3\right] = -\frac{a^2}{6}\left[(1-1)^3 - (1+1)^3\right] = \frac{8a^2}{6} = \frac{4a^2}{3}$$

Problem

Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r dr d\theta$

Sol:Given,

$$\int_0^{\pi}\int_0^{a\sin\theta} r dr d\theta = \int_0^{\pi}[\int_0^{a\sin\theta} r dr]d\theta$$

$$= \int_{\theta=0}^{\pi}[\frac{r^2}{2}]_{r=0}^{a\sin\theta} d\theta = \int_{\theta=0}^{\pi}[\frac{a^2}{2}\sin^2\theta]d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi} \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{a^2}{4} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4} [(\pi - 0) - (0 - 0)] = \frac{a^2 \pi}{4}$$

12. **Evaluate $\iint r^3 dr d\theta$, over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.**

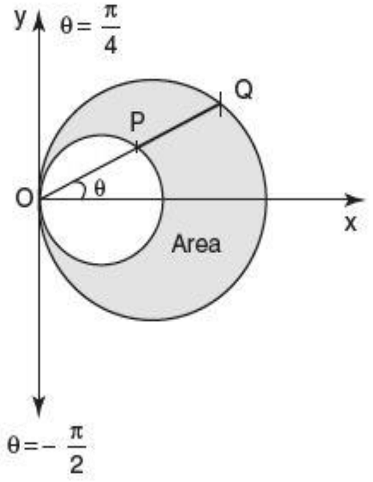
Solution.

Let $$I = \iint_A r^3 dr d\theta,$$

where the region A is the area between the circles

$$r = 2\cos\theta \text{ and } r = 4\cos\theta$$

The area A is the shaded area in the **Fig.**



We first integrate w.r.to r. So, take a radius vector OPQ, where r varies from P to Q.

$\therefore$ r varies from $2\cos\theta$ to $4\cos\theta$

When PQ is varied to cover the area A between

$r = 2\cos\theta$ and $r = 4\cos\theta$, θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\therefore \quad I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r^3 dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4^4\cos^4\theta - 2^4\cos^4\theta) d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (256 - 16)\cos^4\theta \, d\theta$$

$$= \frac{240}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \theta \, d\theta \quad = 60 \times 2 \int_0^{\frac{\pi}{2}} \cos^4 \theta \, d\theta \qquad [\because \cos^4 \theta \text{ is even}]$$

$$= 120 \times \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2} = \frac{45\pi}{2} \qquad \text{[Using formula]}$$

13.

Evaluate $\iint_R \frac{r\,dr\,d\theta}{\sqrt{r^2 + a^2}}$, where R is the area of one loop of the leminiscate $r^2 = a^2 \cos 2\theta$.

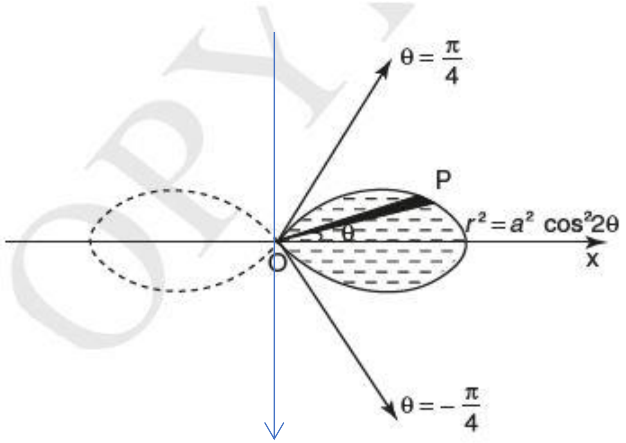
Solution.

Let $I = \iint_R \frac{r\,dr\,d\theta}{\sqrt{r^2 + a^2}}$

First integrate with respect to r

Take a radial strip OP, its ends are $r = 0$ and $r = a\sqrt{\cos 2\theta}$

When the strip covers the region, θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$



$$\therefore \quad I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{r^2+a^2}} dr d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{1}{2} \int_{0}^{a\sqrt{\cos 2\theta}} (r^2+a^2)^{-1/2} 2r dr \right] d\theta$$

$$= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{(r^2+a^2)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} \right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[(r^2+a^2)^{1/2} \right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[(a^2 \cos 2\theta + a^2)^{1/2} - (a^2)^{1/2} \right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \{ a[\cos 2\theta + 1]^{1/2} - a \} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} [a(2\cos^2\theta)^{1/2} - a] d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} a(\sqrt{2}\cos\theta - 1)\,d\theta \qquad [\because \sqrt{2}\cos\theta - 1 \text{ is even function}]$$

$$= 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4}$$

$$= 2a\left\{\left[\sqrt{2}\sin\frac{\pi}{4} - \frac{\pi}{4}\right] - (\sqrt{2}\sin 0 - 0)\right\} = 2a\left[\left(\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right) - 0\right] = 2a\left[1 - \frac{\pi}{4}\right]$$

Change of Variables in Double Integral

The evaluation of a double integral, sometimes become simpler if the variables of integration are transformed suitably into new variables.

For example, from cartesian coordinates to polar coordinates or to some variables u and v.

1. Change of variables from x, y to the variables u and v.

Let $\iint_R f(x, y)\, dxdy$ be the given double integral.

Suppose $x = g(u, v)$, $y = h(u, v)$ be the transformations. Then $dxdy = |J|\, dudv$, where $J = \frac{\partial(x, y)}{\partial(u, v)}$ is the Jacobian of the transformation.

$$\therefore \quad \iint_R f(x, y)\, dxdy = \iint_R F(u, v)\, |J|\, dudv$$

2. Change of variable from Cartesian to polar coordinates

Let $\iint_R f(x, y)\, dxdy$ be the double integral.

Let $x = r\cos\theta$, $y = r\sin\theta$ be the transformation from Cartesian to polar coordinates.

Then $dxdy = |J|\, drd\theta$

where $J = \frac{\partial(x, y)}{\partial(r, \theta)}$ is the Jacobian of transformation.

and
$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

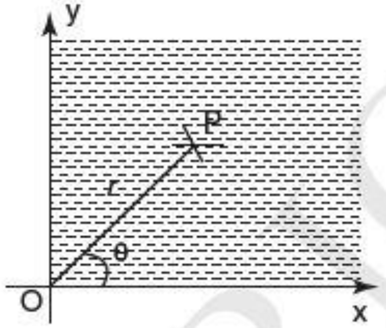
$\therefore \quad dxdy = r\,drd\theta \quad$ and $\quad \therefore \iint_R f(x, y)\, dxdy = \iint_R F(r, \theta)\, r\, drd\theta$

Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy$ by changing to polar coordinates and hence evaluate $\int_0^\infty e^{-x^2}dx$.

Solution.

Let $$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy$$

Since x varies from 0 to ∞ and y varies from 0 to ∞, it is clear that the region of integration is the first quadrant as in Fig. 13.23



To change to polar coordinates, put $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \quad dxdy = rdrd\theta$

and $\quad x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\therefore \quad r$ varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\,rdrd\theta$$

Put $\quad r^2 = t \Rightarrow 2rdr = dt \Rightarrow rdr = \frac{dt}{2}$

When $\quad r = 0, t = 0$ and when $r = \infty, t = \infty$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} \int_0^{\infty} e^{-t} dt \right] d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[\frac{e^{-t}}{-1} \right]_0^{\infty} d\theta = -\frac{1}{2} \int_0^{\frac{\pi}{2}} (e^{-\infty} - e^{0}) d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} (0-1)\, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2} [\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\therefore \quad \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\, dx dy = \frac{\pi}{4}$$

To find $\int e^{-x^2}\, dx$

Now, $\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy = \int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy \quad \Rightarrow \quad \frac{\pi}{4} = \left[\int_0^{\infty} e^{-x^2} dx \right]^2 \qquad \left[\because \int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-y^2} dy \right]$

$$\therefore \quad \int_0^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ **by changing into polar coordinates.**

Solution.

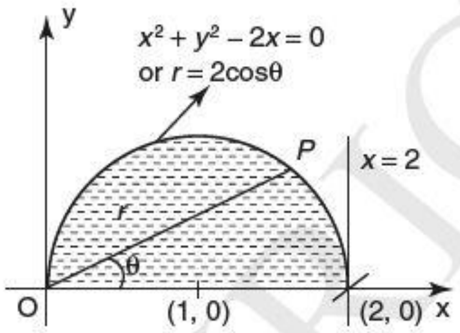
Let $$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$

The limits for y are $y = 0$ and $y = \sqrt{2x - x^2}$

Now, $y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 + y^2 - 2x = 0 \Rightarrow (x-1)^2 + y^2 = 1,$

which is a circle with centre (1, 0) and radius $r = 1$ and x varies from 0 to 2.

$\therefore$ the region of integration is the upper semi-circle as in **Fig. 13.24**



To change to polar coordinates, put $x = r\cos\theta$, $y = r\sin\theta$

$\therefore dx\,dy = r\,dr\,d\theta$

$\therefore$ $$x^2 + y^2 - 2x = 0$$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta = 0$

$\Rightarrow$ $$r^2 - 2r\cos\theta = 0 \Rightarrow r(r - 2\cos\theta) = 0 \Rightarrow r = 0, 2\cos\theta$$

Limits of r are $r = 0$ and $r = 2\cos\theta$ and limits of θ are $\theta = 0$ and $\theta = \frac{\pi}{2}$

$$\therefore \qquad I = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} \frac{r\cos\theta}{r} r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\cos\theta\, dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta \left[\int_0^{2\cos\theta} r\,dr\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos\theta\, 4\cos^2\theta\, d\theta = 2\int_0^{\frac{\pi}{2}} \cos^3 d\theta = 2\cdot\frac{3-1}{3}\cdot 1 = \frac{4}{3}$$

By changing into polar coordinates, evaluate the integral $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$.

Solution.

Let $\quad I = \int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$

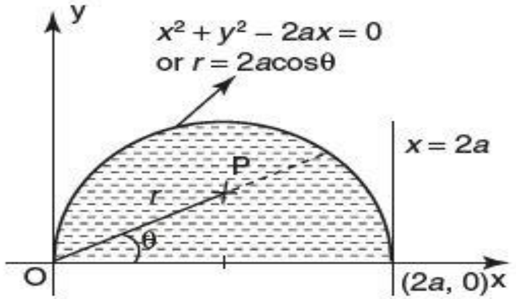
The limits for y are $y = 0$ and $y = \sqrt{2ax - x^2}$

Now, $\quad y = \sqrt{2ax - x^2} \quad \Rightarrow \quad y^2 = 2ax - x^2$

$\Rightarrow x^2 + y^2 - 2ax = 0 \qquad \Rightarrow \quad (x-a)^2 + y^2 = a^2$

which is a circle with centre $(a, 0)$ and radius $r = a$.

$\therefore$ x varies from 0 to $2a$



$\therefore$ the region of integration is the upper semi circle as in **Fig. 13.25.**

To change to polar coordinates, put $x = r\cos\theta$ and $y = r\sin\theta$.

$$\therefore \quad dxdy = rdrd\theta \text{ and } x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

and $x^2 + y^2 - 2ax = 0 \Rightarrow r^2 - 2ar\cos\theta = 0 \Rightarrow r(r - 2a\cos\theta) = 0 \Rightarrow r = 0,\ r = 2a\cos\theta$

$\therefore$ r varies from 0 to $2a\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2 \cdot r\,drd\theta = \int_0^{\frac{\pi}{2}}\left[\int_0^{2a\cos\theta} r^3 dr\right]d\theta$$

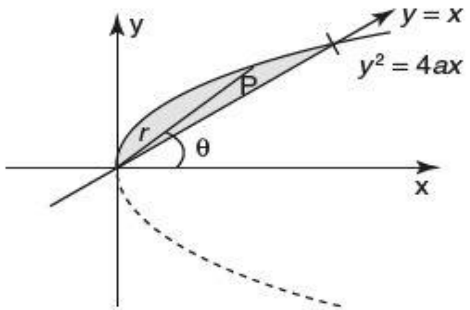
$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}}(2a)^4\frac{\cos^4\theta}{4}d\theta = \frac{16a^4}{4}\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta = 4a^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3a^4\pi}{4}$$

Evaluate $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}dxdy$ **by changing to polar coordinates.**

Solution.

Let $$I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}dxdy$$



Given, the limits for x are $x = \frac{y^2}{4a}$ and $x = y$

$\Rightarrow$ $y^2 = 4ax$ and $y = x$

And the limits for y are $y = 0$ and $y = 4a$

To find the point of intersection of $y^2 = 4ax$ and $y = x$,

solve the two equations.

Now $y^2 = 4ax \Rightarrow y^2 = 4ay \Rightarrow y(y-4a) = 0 \Rightarrow y = 0, \quad y = 4a$

$\therefore$ $x = 0, \quad x = 4a$

$\therefore$ the points are $(0, 0)$, $(4a, 4a)$

$\therefore$ the region of integration is the shaded region as in **Fig. 13.26** which is bounded by $y^2 = 4ax$ and $y = x$.

To change to polar coordinates, put $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $dxdy = rdrd\theta$ and $x^2 + y^2 = r^2$

$\therefore$ $x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$

and $y^2 = 4ax$ becomes $r^2\sin^2\theta = 4a \cdot r\cos\theta \Rightarrow r(r\sin^2\theta - 4a\cos\theta) = 0$

$$\Rightarrow \quad r = 0 \text{ and } r\sin^2\theta - 4a\cos\theta = 0 \quad \Rightarrow \quad r = \frac{4a\cos\theta}{\sin^2\theta}$$

$\therefore$ limits for r are $0, \dfrac{4a\cos\theta}{\sin^2\theta}$ and θ varies from $\dfrac{\pi}{4}$ to $\dfrac{\pi}{2}$. $\quad [\because$ slope of the line is $\tan\theta = 1 \Rightarrow \theta = \dfrac{\pi}{4}]$

$$\therefore \quad I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{r^2\cos 2\theta}{r^2} r\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2\theta \left[\int_0^{\frac{4a\cos\theta}{\sin^2\theta}} r\,dr\right] d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2\theta \left[\frac{r^2}{2}\right]_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2\theta \frac{16a^2 \times \cos^2\theta}{\sin^4\theta} d\theta$$

$$= \frac{16a^2}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos^2\theta - \sin^2\theta)\frac{\cos^2\theta}{\sin^4\theta} d\theta$$

$$= 8a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{\cos^2 \theta}{\sin^2 \theta} - 1 \right) \frac{\sin^2 \theta \cos^2 \theta}{\sin^4 \theta} d\theta$$

$$= 8a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^2 \theta - 1) \frac{\cos^2 \theta}{\sin^2 \theta} d\theta$$

$$= 8a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\operatorname{cosec}^2\theta - 1 - 1) \cot^2 \theta d\theta$$

$$= 8a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\operatorname{cosec}^2\theta - 2) \cot^2 \theta \, d\theta$$

$$= 8a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\operatorname{cosec}^2\theta \cot^2 \theta - 2\cot^2 \theta) \, d\theta$$

$$= 8a^2 \left[\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{cosec}^2\theta \cot^2 \theta d\theta - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2 \theta d\theta \right]$$

$$= 8a^2\left[\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2\theta \,\text{cosec}^2\theta\, d\theta - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\text{cosec}^2\theta - 1)\, d\theta\right]$$

$$= 8a^2\left[\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\cot^2\theta\,(-\text{cosec}^2\theta)\, d\theta - 2[-\cot\theta - \theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

$$= 8a^2\left[-\frac{1}{3}[\cot^3\theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + 2\left\{\cot\frac{\pi}{2} + \frac{\pi}{2} - \left(\cot\frac{\pi}{4} + \frac{\pi}{4}\right)\right\}\right]$$

$$= 8a^2\left[-\frac{1}{3}\left(\cot^3\frac{\pi}{2} - \cot^3\frac{\pi}{4}\right) + 2\left(0 + \frac{\pi}{2} - 1 - \frac{\pi}{4}\right)\right]$$

$$= 8a^2\left[-\frac{1}{3}(-1) - 2 + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right] = 8a^2\left[\frac{1}{3} - 2 + \frac{\pi}{2}\right] = \frac{8a^2}{6}(3\pi - 10) = \frac{4a^2}{3}(3\pi - 10)$$

Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dxdy$ by changing into polar coordinates.

Solution.

Let $$I = \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dxdy$$

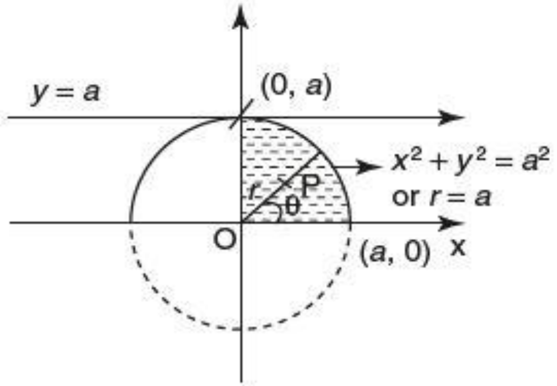
Limits for x are $x = 0$ and $x = \sqrt{a^2-y^2}$

Now $$x = \sqrt{a^2-y^2} \Rightarrow x^2 = a^2 - y^2 \Rightarrow x^2 + y^2 = a^2$$

which is circle with centre (0, 0) and radius a

Limits for y are $y = 0$ and $y = a$

$\therefore$ the region of integration is as in **Fig.**



bounded by $y = 0$, $y = a$ and $x = 0$, $x = \sqrt{a^2 - y^2}$

To change to polar coordinates,

put $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \quad dxdy = rdrd\theta \quad \text{and} \quad x^2 + y^2 = r^2$$

$$\therefore \quad x^2 + y^2 = a^2 \quad \Rightarrow \quad r^2 = a^2 \quad \Rightarrow \quad r = \pm a$$

$\therefore$ in the given region, r varies from 0 to a and θ varies from 0 to $\frac{\pi}{2}$

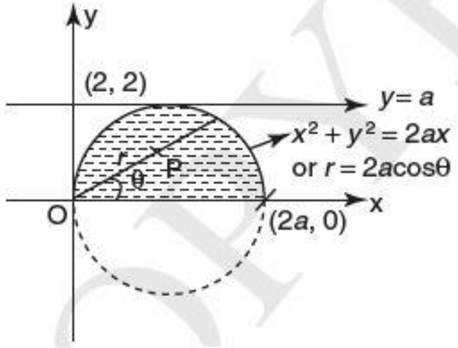
$$\therefore \text{ I} = \int_0^{\frac{\pi}{2}}\int_0^{a} r^2 \cdot rdrd\theta = \int_0^{\frac{\pi}{2}}\left[\int_0^{a} r^3 dr\right]d\theta = \int_0^{\frac{\pi}{4}}\left[\frac{r^4}{4}\right]_0^a d\theta = \int_0^{\frac{\pi}{4}} \frac{a^4}{4}\, d\theta = \frac{a^4}{4}[\theta]_0^{\frac{\pi}{2}} = \frac{\pi a^4}{8}.$$

Evaluate $\iint_R y\,dx\,dy$, where R is the region bounded by the semi-circle $x^2 + y^2 = 2ax$ and the x-axis and the lines $y = 0$ and $y = a$.

Solution.

Let $$I = \iint_R y\,dx\,dy$$

The region R is as in **Fig**.



We have $x^2 + y^2 = 2ax$

$\Rightarrow \quad x^2 - 2ax + y^2 = 0$

$\Rightarrow \quad (x-a)^2 + y^2 = a^2$

which is a circle with centre $(a, 0)$ and radius a

To change to polar coordinates, put $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \quad dxdy = r\,dr\,d\theta$ and $x^2 + y^2 = r^2$

Now $x^2 + y^2 = 2ax \Rightarrow r^2 = 2ar\cos\theta$

$\Rightarrow r^2 - 2ar\cos\theta = 0 \Rightarrow r(r - 2a\cos\theta) = 0 \Rightarrow r = 0,\ r = 2a\cos\theta$

$\therefore$ r varies from 0 to $2a\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$

$$\therefore\ I = \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r\sin\theta \cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \sin\theta \left[r^2 \int_0^{2a\cos\theta} dr \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \left[\frac{r^3}{3} \right]^{2a\cos\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} \sin\theta (2a)^3 \cos^3\theta\, d\theta$$

$$= \frac{8a^3}{3} \int_0^{\frac{\pi}{2}} \cos^3\theta \sin\theta\, d\theta$$

$$= \frac{8a^3}{3} \left[\frac{-\cos^4\theta}{4} \right]_0^{\frac{\pi}{2}} = -\frac{2a^3}{3} \left[\cos^4\frac{\pi}{2} - \cos 0 \right] = -\frac{2a^3}{3}(0 - 1) = \frac{2a^3}{3}$$

TRIPLE INTEGRAL IN CARTESIAN COORDINATES

Let $f(x, y, z)$ be a continuous function at every point in a closed and bounded region D in space. Subdivide the region into a number of element volumes by drawing planes parallel to the coordinate planes. Let $\Delta V_1, \Delta V_2, \ldots, \Delta V_n$ be the number of element volumes formed. Let (x_i, y_i, z_i) be any point in ΔV_i, where $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$. Form the sum $\sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta V_i$. The limit of the sum as $n \to \infty$ and $\Delta V_i \to 0$, if it exists, is called the **triple integral** of $f(x, y, z)$ over D and is denoted by

$$\iiint_D f(x,y,z)\, dV \quad \text{or} \quad \iiint_D f(x,y,z)\, dxdydz \tag{1}$$

As in the case of double integrals, the triple integral is evaluated by three successive integration of single variable.

Consider the triple integral

$$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)\, dxdydz$$

(1) If all the limits are constants, then the integration can be performed in any order with proper limits,

i.e., $$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)\, dxdydz = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} f(x,y,z)\, dzdydx = \int_{x_0}^{x_1}\int_{z_0}^{z_1}\int_{y_0}^{y_1} f(x,y,z)\, dydzdx$$

(2) If $x_0 = f_0(y,z)$, $x_1 = f_1(y,z)$, $y_0 = g_0(z)$, $y_1 = g_1(z)$, $z_0 = a$, $z_1 = b$,

then $$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)\,dx\,dy\,dz = \int_a^b \left[\int_{y_0=g_0(z)}^{y_1=g_1(z)} \left[\int_{x_0=f_0(y,z)}^{x_1=f_1(y,z)} f(x,y,z)\,dx\right] dy\right] dz$$

First we integrate w.r.to x, treating y and z as constants and substitute limits of x. Next integrate the resulting function of y and z w.r.to y, treating z as constant and substitute the limits of y. Finally we integrate the resulting function of z w.r.to z and substitute the limits of z.

9. **Evaluate** $\int_0^1\int_0^2\int_1^2 x^2yz\,dx\,dy\,dz$.

Solution.

Let $I = \int_0^1\int_0^2\int_1^2 x^2yz\,dx\,dy\,dz$

$\Rightarrow I = \int_0^1 z\,dz \int_0^2 y\,dy \int_1^2 x^2\,dx = \left(\frac{z^2}{2}\right)_0^1 \left(\frac{y^2}{2}\right)_0^2 \left(\frac{x^3}{3}\right)_1^2 = \frac{1}{2}\cdot\frac{4}{2}\left(\frac{8}{3}-\frac{1}{3}\right) = \frac{7}{3}$ [$\because$ limits are constants]

10. Evaluate $\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dxdydz$.

Solution.

Let $I=\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dxdydz=\int_0^a\int_0^b\left[\frac{x^3}{3}+(y^2+z^2)x\right]_0^c dydz$

$$=\int_0^a\int_0^b\left[\frac{c^3}{3}+(y^2+z^2)\,c\right]dydz$$

$$=c\int_0^a\int_0^b\left(\frac{c^2}{3}+y^2+z^2\right)dydz$$

$$=c\int_0^a\left[\frac{c^2}{3}y+\frac{y^3}{3}+z^2y\right]_0^b dz$$

$$=c\int_0^a\left[\frac{c^2b}{3}+\frac{b^3}{3}+z^2b\right]dz$$

$$=bc\int_0^a\left[\frac{c^2}{3}+\frac{b^2}{3}+z^2\right]dz$$

$$=bc\left[\left(\frac{c^2}{3}+\frac{b^2}{3}\right)z+\frac{z^3}{3}\right]_0^a=bc\left[\left(\frac{b^2+c^2}{3}\right)a+\frac{a^3}{3}\right]=abc\left[\frac{a^2+b^2+c^2}{3}\right]$$

11. **Evaluate** $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{(x+y+z)}dxdydz.$

Solution.

$$\text{Let } I=\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{(x+y+z)}dzdydx=\int_0^{\log 2} e^{x+y}.[e^z]^{x+y} dydz=\int_0^{\log 2} e^{x+y}.(e^{x+y}-1)dydz$$

$$=\int_0^{\log 2}\int_0^{x}(e^{(2x+2y)}-e^{x+y})dydz=\int_0^{\log 2}\left\{e^{2x}.\left[\frac{e^{2y}}{2}\right]_0^x-e^x.[e^y]_0^x\right\}dz$$

$$=\frac{1}{2}\int_0^{\log 2}\left[e^{2x}(e^{2x}-1)-2e^x(e^x-1)\right]dz=\frac{1}{2}\int_0^{\log 2}\left[e^{4x}-e^{2x}-2e^{2x}+2e^x\right]dx$$

$$=\frac{1}{2}\int_0^{\log 2}(e^{4x}-3e^{2x}+2e^{x})dx$$

$$=\frac{1}{2}\left[\frac{e^{4x}}{4}-3\frac{e^{2x}}{2}+2e^x\right]_0^{\log_e 2}$$

$$=\frac{1}{2}\left[\left(\frac{e^{4\log_e 2}}{4}-\frac{3}{2}e^{2\log_e 2}+2e^{\log_e 2}\right)-\left(\frac{1}{4}-\frac{3}{2}+2\right)\right]$$

$$=\frac{1}{2}\left[\frac{e^{\log_e 16}}{4}-\frac{3}{2}e^{\log_e 4}+2e^{\log_e 2}-\frac{3}{4}\right]=\frac{1}{2}\left[\frac{16}{4}-\frac{3\cdot 4}{2}+2\cdot 2-\frac{3}{4}\right]=2-3+2-\frac{3}{8}=1-\frac{3}{8}=\frac{5}{8}$$

$$[\because e^{\log_e x}=x]$$

12. **Evaluate** $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\,dx\,dz.$

Solution.

Let $$I = \int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\,dx\,dz = \int_0^4 \int_0^{2\sqrt{z}} [y]_0^{\sqrt{4z-x^2}} dx\,dz \qquad \text{[Treating } x, z \text{ constants]}$$

$$= \int_0^4 \int_0^{2\sqrt{z}} \sqrt{4z-x^2}\,dx\,dz$$

$$= \int_0^4 \left[\frac{x}{2}\sqrt{4z-x^2} + \frac{4z}{2}\sin^{-1}\frac{x}{2\sqrt{z}} \right]_0^{2\sqrt{z}} dz \qquad \text{[Treating } z \text{ constant]}$$

$$= \int_0^4 [\sqrt{z}\sqrt{4z-4z} + 2z\sin^{-1}1 - 0]\,dz$$

$$= \int_0^4 2z\frac{\pi}{2}dz = \pi\int_0^4 z\,dz = \pi\left[\frac{z^2}{2}\right]_0^4 = \frac{\pi}{2}(16) = 8\pi$$

13. **Evaluate** $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log 2} e^{x+y+z}\, dzdydx.$

Solution.

Let $I = \int_0^{\log 2}\int_0^{x}\int_0^{x+\log_e y} e^{x+y+z} dzdydx = \int_0^{\log 2}\int_0^{x} e^x \cdot e^y [e^z]_0^{x+\log y} dydx$

$$= \int_0^{\log 2}\int_0^{x} e^x \cdot e^y [e^{x+\log y} - e^0]\, dydx$$

$$= \int_0^{\log 2}\int_0^{x} e^x e^y (e^x \cdot e^{\log y} - 1) dydx$$

$$= \int_0^{\log 2}\int_0^{x} e^x e^y (e^x \cdot y - 1)\, dydx \qquad [\because\ e^{\log_e y} = y]$$

$$= \int_0^{\log 2}\int_0^{x} (e^{2x} \cdot ye^y - e^x \cdot e^y)\, dydx$$

$$= \int_0^{\log 2}\left[e^{2x}\int_0^{x} ye^y dy - e^x \int_0^{x} e^y dy \right] dx$$

$$= \int_0^{\log 2} \left[e^{2x} \int_0^x y e^y \, dy - e^x \int_0^x e^y \, dy \right] dx$$

$$= \int_0^{\log 2} \{ e^{2x} [y \cdot e^y - 1 \cdot e^y]_0^x - e^x [e^y]_0^x \} dx \qquad \text{[Using Bernoulli's formula]}$$

$$= \int_0^{\log 2} \{ e^{2x} [x e^x - e^x - (0-1)] - e^x (e^x - 1) \} dx$$

$$= \int_0^{\log 2} \{ (x-1) e^{3x} + e^{2x} - e^{2x} + e^x \} dx$$

$$= \int_0^{\log 2} \{ (x-1) e^{3x} + e^x \} dx$$

$$= \left[(x-1) \frac{e^{3x}}{3} - 1 \cdot \frac{e^{3x}}{9} + e^x \right]_0^{\log_e 2}$$

$$= \left\{ \frac{1}{3} (\log_e 2 - 1) e^{3 \log_e 2} - \frac{1}{9} e^{3 \log_e 2} + e^{\log_e 2} - \left[-\frac{1}{3} - \frac{1}{9} + 1 \right] \right\}$$

$$= \frac{1}{3} (\log_e 2 - 1) \cdot 8 - \frac{8}{9} + 2 - \frac{5}{9} \qquad [\because \; e^{3 \log_e 2} = e^{\log_e 2^3} = 2^3 = 8 \text{ and } e^{\log_e 2} = 2]$$

$$= \frac{8}{3} \log_e 2 - \frac{8}{3} - \frac{8}{9} + 2 - \frac{5}{9} = \frac{8}{3} \log_e 2 - \frac{19}{9} = \frac{1}{9} (24 \log 2 - 19)$$

14. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{a^2-x^2-y^2-z^2}}$.

Solution.

Let $$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{(a^2-x^2-y^2)-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dydx \qquad \left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}\right]$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} [\sin^{-1}1 - \sin^{-1}0]\, dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2} dydx$$

$$= \frac{\pi}{2}\int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\pi}{2}\int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = \frac{\pi}{2}\left[0 + \frac{a^2}{2}\sin^{-1}1 - 0\right] = \frac{\pi}{2}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi^2 a^2}{8}$$

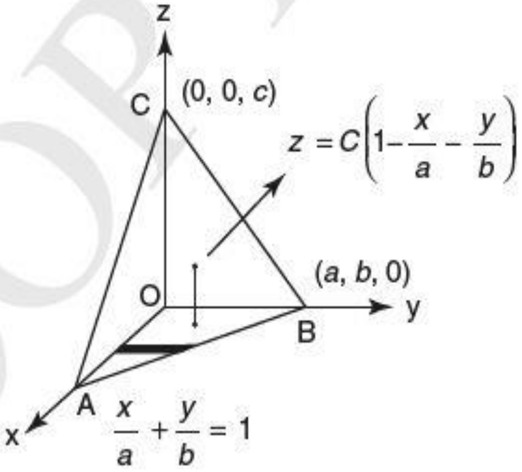
15. Evaluate $\iiint_V xyz\, dxdydz$ over the volume V enclosed by the three coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Solution.

Let V be the volume enclosed by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and it meets the coordinate axes in A(a, 0, 0), B(0, b, 0), C(0, 0, c) respectively.

The projection of V on the xy-plane is the ΔOAB

bounded by $x=0, y=0, \frac{x}{a}+\frac{y}{b}=1$



z varies from 0 to $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

y varies from 0 to $b\left(1-\frac{x}{a}\right)$

and x varies from 0 to a.

$$I=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} xyz\,dzdydx$$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} xy\left[\frac{z^2}{2}\right]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$=\frac{1}{2}\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} xyc^2\left(1-\frac{x}{a}-\frac{y}{b}\right)^2 dydx$$

$$=\frac{c^2}{2}\int_0^a x\int_0^{b\left(1-\frac{x}{a}\right)} y\left[\left(1-\frac{x}{a}\right)-\frac{y}{b}\right]^2 dydx$$

$$=\frac{c^2}{2}\int_0^a x\left[\frac{y\left\{\left(1-\frac{x}{a}\right)-\frac{y}{b}\right\}^3}{-\frac{1}{b}\cdot 3}-1\cdot\frac{\left\{\left(1-\frac{x}{a}\right)-\frac{y}{b}\right\}^4}{\frac{-3}{b}\cdot\left(\frac{-1}{b}\cdot 4\right)}\right]_0^{b\left(1-\frac{x}{a}\right)} dx \qquad \text{[Using Bernouli's formula]}$$

$$= \frac{c^2}{2}\int_0^a x\left[\frac{b}{3}\left(1-\frac{x}{a}\right)(0)-0-\frac{b^2}{12}\left(0-\left(1-\frac{x}{a}\right)^4\right)\right]dx$$

$$= \frac{c^2 \cdot b^2}{24}\int_0^a x\left(1-\frac{x}{a}\right)^4 dx$$

$$= \frac{b^2c^2}{24}\left[x\cdot\frac{\left(1-\frac{x}{a}\right)^5}{-\frac{1}{a}\cdot 5}-1\cdot\frac{\left(1-\frac{x}{a}\right)^6}{-\frac{5}{a}\cdot\left(\frac{-6}{a}\right)}\right]_0^a = \frac{b^2c^2}{24}\left[0-\frac{a^2}{30}(0-1)\right] = \frac{b^2c^2}{24}\cdot\frac{a^2}{30} = \frac{a^2b^2c^2}{720}$$

Evaluation of triple integrals when region is given:-

① Sphere $x^2+y^2+z^2=a^2$

$z^2 = a^2-x^2-y^2$ $\quad \therefore z = \pm\sqrt{a^2-x^2-y^2}$ ✓ $z: -\sqrt{a^2-x^2-y^2} \to \sqrt{a^2-x^2-y^2}$

$z=0,\ x^2+y^2=a^2 \Rightarrow y^2=a^2-x^2$

$y = \pm\sqrt{a^2-x^2}$ ✓ $y: -\sqrt{a^2-x^2} \to \sqrt{a^2-x^2}$

$y=z=0,\ x^2=a^2 \Rightarrow x=\pm a$ $\quad \therefore$ ✓ $x: -a \to a$

R is the sphere in the first quadrant:-

$x: 0 \to a$

$y: 0 \to \sqrt{a^2-x^2}$

$z: 0 \to \sqrt{a^2-x^2-y^2}$

$x: -\sqrt{a^2-y^2-z^2} \to \sqrt{a^2-y^2-z^2}$

$y: -\sqrt{a^2-z^2} \to \sqrt{a^2-z^2}$

$z: -a \to a$

$z \to 0 \to a$

$x: 0 \to \sqrt{a^2-y^2-z^2}$

$y: 0 \to \sqrt{a^2-z^2}$

} first quadrant

(2) plane:

$2x+3y+2z=1$

$2x = 1-3y-2z$

$x = \frac{1}{2}(1-3y-2z)$

$x=0,\quad 3y+2z=1$

$3y = 1-2z$

$y = \frac{1-2z}{3}$

$x=y=0,\quad 2z=1$

$\Rightarrow z=\frac{1}{2}$

first quadrant

$x: 0 \rightarrow (1-3y-2z)/2$

$y: 0 \rightarrow (1-2z)/3,\quad z: 0 \rightarrow 1/2$

(or) $2z = 1-2x-3y$

$z = (1-2x-3y)/2$

$z=0,\quad 2x+3y=1$

$3y = 1-2x$

$y = (1-2x)/3$

$y=z=0,\quad 2x=1$

$\Rightarrow x=\frac{1}{2}$

first quadrant,

$x: 0 \rightarrow \frac{1}{2}$

$y: 0 \rightarrow (1-2x)/3$

$z: 0 \rightarrow (1-2x-3y)/2$

3) Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$z^2=c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$z: -c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \rightarrow c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$z=0$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$\Rightarrow y^2=b^2\sqrt{1-\frac{x^2}{a^2}} \Rightarrow y=\pm b\sqrt{1-\frac{x^2}{a^2}}$

$y: -b\sqrt{1-\frac{x^2}{a^2}} \rightarrow b\sqrt{1-\frac{x^2}{a^2}}$

$y=0, z=0 \Rightarrow \frac{x^2}{a^2}=1$

$x^2=a^2 \Rightarrow x=\pm a$

$x: -a \rightarrow a$

In first or positive quadrant:-

$x: 0 \rightarrow a$

$y: 0 \rightarrow b\sqrt{1-\frac{x^2}{a^2}}$

$z: 0 \rightarrow c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

Change of variables in triple integrals :-

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(u,v,w)\,|J|\,du\,dv\,dw$$

Change of variables from cartesian to spherical polar coordinates :-

$$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$|J| = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Change of variables from cartesian to cylindrical coordinates :-

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$dx\,dy\,dz = r\,dr\,d\theta\,d\phi \qquad |J| = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r.$$

$x = r\sin\theta\cos\phi$

$x_r = \sin\theta\cos\phi$

$x_\theta = r\cos\theta\cos\phi$

$x_\phi = -r\sin\theta\sin\phi$

$y = r\sin\theta\sin\phi$

$y_r = \sin\theta\sin\phi$

$y_\theta = r\cos\theta\sin\phi$

$y_\phi = r\sin\theta\cos\phi$

$z = r\cos\theta$

$z_r = \cos\theta$

$z_\theta = -r\sin\theta$

$z_\phi = 0$

$$|J| = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right] + r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right] = \cos\theta \cdot r^2\sin\theta\cos\theta(1) + r\sin^2\theta\cdot(1)\, r\sin\theta$$

$$= r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta$$

Change to spherical polar coordinates and hence evaluate $\iiint_V \frac{dxdydz}{x^2+y^2+z^2}$ where V is the volume of the sphere $x^2+y^2+z^2=a^2$.

Solution.

$$I = \iiint_V \frac{1}{x^2+y^2+z^2}\,dxdydz$$

Using spherical polar coordinates (r, θ, ϕ), $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Then the Jacobian of transformation is

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

$$\therefore \quad dxdydz = |J|\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$x^2+y^2+z^2 = r^2\sin^2\cos^2\phi + r\sin^2\theta\sin^2\phi + r^2\cos^2\theta$$

$$= r^2\sin^2\theta[\cos^2\phi+\sin^2\phi] + r^2\cos^2\theta = r^2[\sin^2\theta+\cos^2\theta] = r^2$$

$$Required = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \frac{r^2 \sin\theta \, dr d\theta d\phi}{r^2}$$

$$= 4\pi a$$

$Evaluate \iiint z(x^2 + y^2)\, dxdydz$ over the volume of the cylinder $x^2 + y^2 = 1$ bounded between $z = 2$ and $z = 3$ by cylindrical polar coordinates

$$Sol := \iiint z(x^2 + y^2)\, dxdydz$$

changing cylindrical polar coordinates

by taking $x = r\cos\theta, y = r\sin\theta, z = z$ and

$$dxdydz = r dr d\theta dz$$

$$= \int_{z=2}^{z=3}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} zr^2 \, r dr d\theta dz$$

$$= 5\pi / 4$$

Applications of Multiple integrals:

Area using double integrals:

Consider the area enclosed by the curve y = f(x), y =g(x), x=a, x=b in the xy- plane.

The area of the region R bounded by the given curve is given by

$$A = \iint_R dx\,dy \text{ or } \iint_R dy\,dx = \int_{x=a}^{b} \int_{y=f(x)}^{g(x)} dy\,dx$$

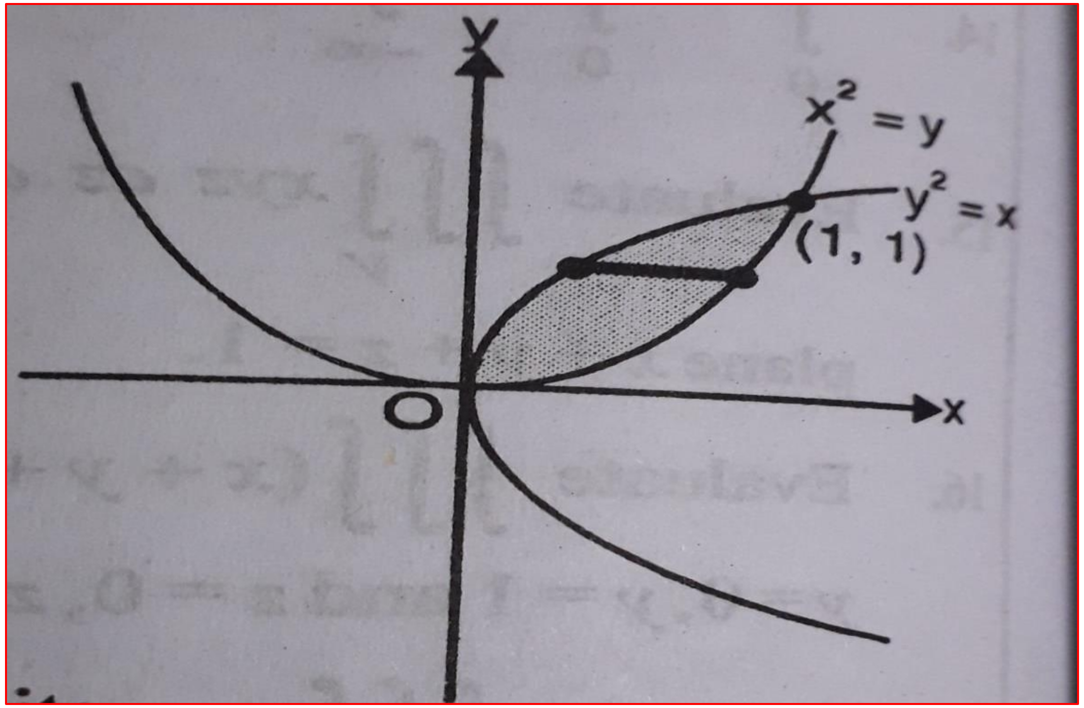
If the region is represented through polar coordinates, then the area is given by

$$A = \iint_R r\,dr\,d\theta$$

1) Evaluate the area enclosed by the parabolas $x^2 = y \; and \; y^2 = x$

Sol:

Given curves are $x^2 = y \; and \; y^2 = x$



$$x^2 = y$$
$$=> x^4 = y^2 = x$$
$$x^4 - x = 0$$
$x(x^3 - 1) = 0 =>$ x=0 , x=1

Hence the intersection points are (0,0) and (1,1)

Y limits 0 to 1

X limits y^2 to $\sqrt{y}$

The required area $A = \iint_R dx\, dy$

$$= \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} dx\, dy$$

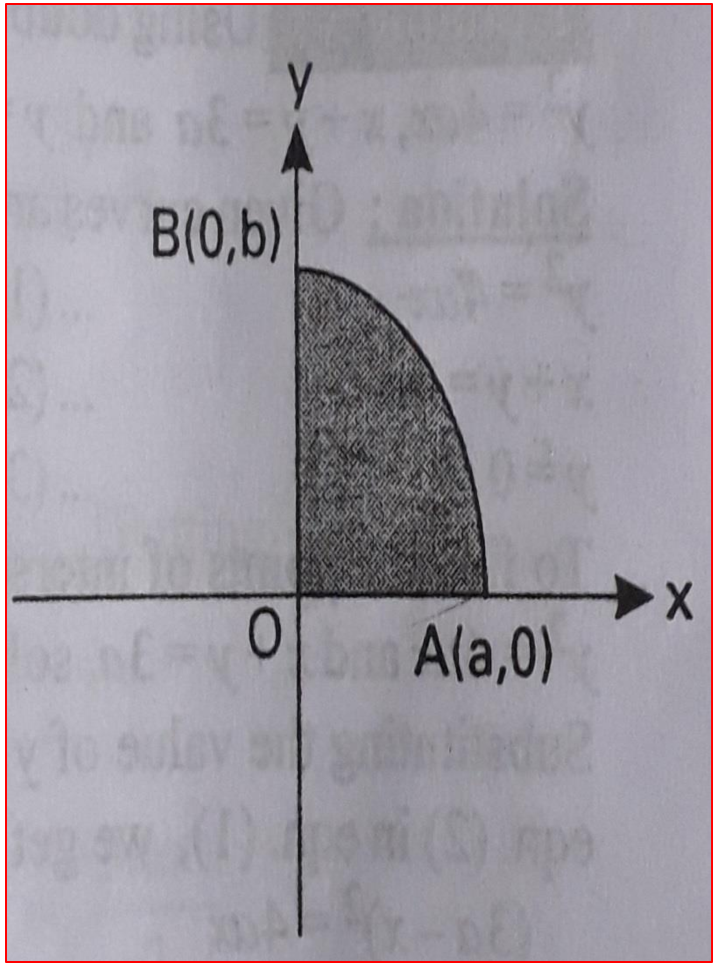
$$= \int_{y=0}^{1} \left(\sqrt{y} - y^2\right) dy$$

$$= \left[\frac{y^{3/2}}{3/2} - \frac{y^3}{3}\right]_{y=0}^{1}$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

2) Find the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol: Given curve is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



X limits 0 to a

Y limits 0 to $\frac{b}{a}\sqrt{a^2 - x^2}$

Area= $A = \iint_R dx\, dy$

$$= \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\, dx$$

$$= \int_{y=0}^{1} \left(\frac{b}{a}\sqrt{a^2 - x^2} - 0\right) dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{b}{a}\left[\frac{a^2}{2}\sin^{-1} 1\right]$$

$$= \frac{ab}{2}.\frac{\pi}{2} = \frac{\pi ab}{4}$$

3)Using double integrals, find the area of the cardioid $r = a(1 - cos\theta)$

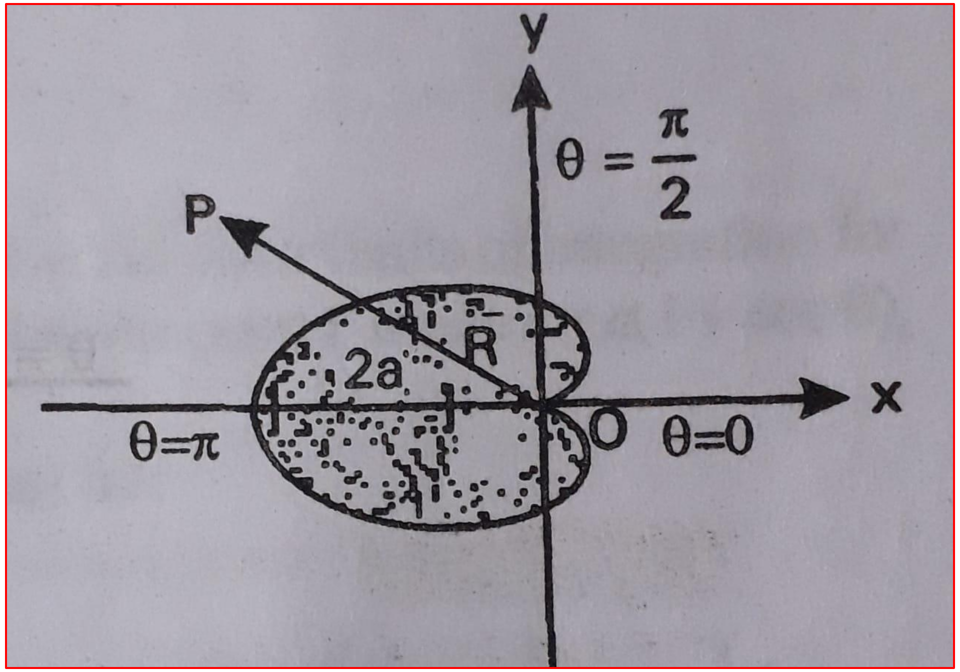
Sol: From the graph of the cardioid $r = a(1 - cos\theta)$

r limits 0 to $a(1 - cos\theta)$

θ limits 0 to π

Area= $A = \iint_R r\, dr\, d\theta$

$$= 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1-cos\theta)} r\, dr\, d\theta$$



$$= 2\int_0^{\pi}\left(\frac{r^2}{2}\right) d\theta$$

$$= \int_0^{\pi}[a(1 - cos\theta)]^2\, d\theta$$

$$= 4a^2 \int_0^{\pi} sin^4 \frac{\theta}{2}\, d\theta$$

$$= 4a^2 \int_0^{\pi/2} sin^4 \emptyset\, 2d\emptyset \text{ (since } \frac{\theta}{2} = \emptyset)$$

$$= 8a^2 . \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$= \frac{3\pi a^2}{2}$$

4) Find the area which is inside the cardioid $r = a(1 + cos\theta)$ and outside the circle r=a.

Sol: From the graph θ $varies\ from$ $\frac{-\pi}{2}$ to $\frac{\pi}{2}$

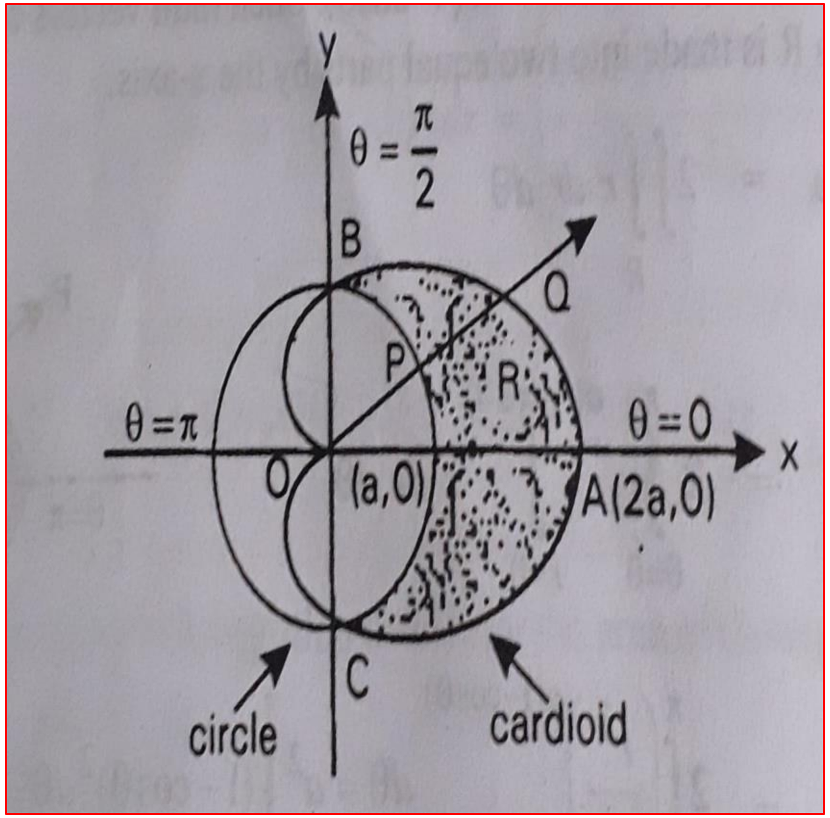
r varies from a to $a(1 + cos\theta)$

Area= $A = \iint_R r\,dr\,d\theta$

$$= 2 \int_{\theta=0}^{\pi/2} \int_{r=a}^{a(1+cos\theta)} r\,dr\,d\theta$$

$$= 2 \int_0^{\pi/2} \left(\frac{r^2}{2}\right) d\theta$$

$$= \int_0^{\pi/2} [a(1 + cos\theta)]^2 - a^2\, d\theta$$



$$=a^2\int_0^{\pi/2}[(1+cos\theta)]^2-1\ \ d\theta$$

$$=a^2\int_0^{\pi/2}(2cos\theta+cos^2\theta)d\theta$$

$$=a^2\int_0^{\pi/2}\left(2cos\theta+\frac{1+cos\theta}{2}\right)d\theta$$

$$=a^2\left[2sin\theta+\frac{1}{2}\left(\theta+\frac{sin2\theta}{2}\right)\right]$$

$$=a^2\left[2+\frac{1}{2}\left(\frac{\pi}{2}+0-0-0\right)\right]$$

$$=a^2\left[2+\left(\frac{\pi}{4}\right)\right]=\frac{a^2}{4}(\pi+8)$$

Volume using double integral:

Let z= f(x,y) be a surface above the xy-plane. Then the volume is

$$V = \iint z \, dx \, dy$$

Note: In polar coordinates volume is

$$V = \iint f(r, \theta) \, r dr \, d\theta$$

1) Find the volume bounded by the cylinder $x^2 + y^2 = 4, y + z = 4\ and\ z = 0.$

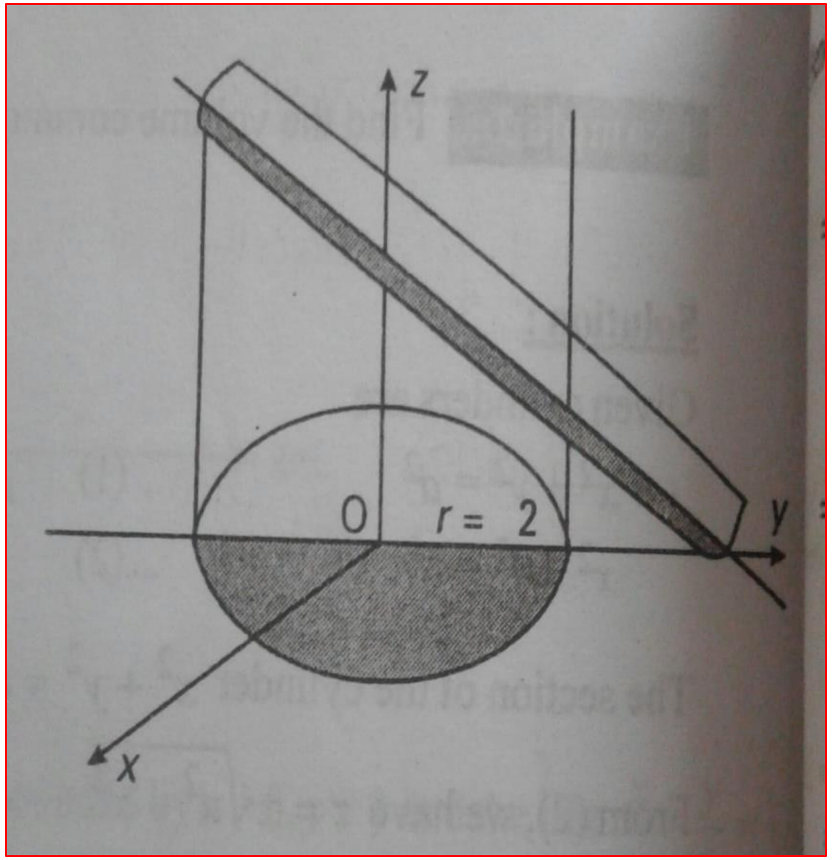
Sol: Equation of the cylinder is $x^2 + y^2 = 4, z = 0$

$x = \pm\sqrt{4 - y^2}$

X=0 then y=$\pm$2

And the equation of the plane is y + z= 4

z = 4-y



X limits $-\sqrt{4 - y^2}$ to $\sqrt{4 - y^2}$

Y limits -2 to 2

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Required volume = $V = \iint z \, dx \, dy$

$$= \int_{y=-2}^{2} \int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4-y) dx \, dy$$

$$= 2 \int_{y=-2}^{2} \int_{x=0}^{\sqrt{4-y^2}} (4-y) dx \, dy$$

$$= 2 \int_{y=-2}^{2} (4-y)[x] \, dy$$

$$= 2 \int_{y=-2}^{2} (4-y) \sqrt{4-y^2} dy$$

$$= 2\int_{y=-2}^{2} (4\sqrt{4-y^2} - y\sqrt{4-y^2})\,dy$$

$$= 8\int_{y=-2}^{2} (\sqrt{4-y^2})\,dy$$

$$= 8(2)\int_{y=0}^{2} (\sqrt{4-y^2})\,dy$$

$$= 16\left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2}\sin^{-1}(\frac{y}{2})\right]$$

$$= 16[0 + 2\sin^{-1}1 - 0]$$

$$= 32 \times \frac{\pi}{2} = 16\pi$$

PRATICE PROBLEM

1) Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

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